Commun Nonlinear Sci Numer Simulat 27 (2015) 294–301

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Motion patterns and phase-transition of a defender–intruder problem and optimal interception strategy of the defender

Jiangliu Wang, Wei Li *

Department of Control and Systems Engineering, Nanjing University, Nanjing, China

ARTICLE INFO

Article history:
Received 5 August 2014
Received in revised form 12 March 2015
Accepted 25 March 2015
Available online 2 April 2015

Keywords:
Nonlinear dynamics
Defense–intrusion
Motion pattern
Pursuit–evasion
Phase-transition
Self-propelled agents

ABSTRACT

In this paper, we consider a defense–intrusion interaction, in which an intruder is attracted by a protected stationary target but repulsed by a defender; while the defender tries to move towards an appropriate interception position (IP) between the intruder and the target in order to intercept the intruder and expel the intruder away from the target as maximum as possible. Intuitionally, to keep the intruder further away, one may wonder that: *is it a better strategy for the defender trying to approach the intruder as near as possible?* Unexpectedly and interestingly enough, this is not always the case. We first introduce the flexibility for IP selection, then investigate the system dynamics and the stable motion patterns, and characterize the phase-transition surface for the motion patterns. We show that, the phase-transition surface just defines the optimal interception strategy of the defender for IP selection; and from the perspective of mobility of agents, the optimal strategy just depends on relative mobility of the two agents.

1. Introduction

Interaction of self-propelled agents has attracted much interest from different disciplines [1–14]. From the very start, the focus of collective motion is on pure coordinated interactions; we call them *non-conflicting* interactions, since the agents have the same role or different but non-conflicting roles (e.g., leaders and followers [4]) with non-conflicting behaviors. In recently years, there is an emerging attention on introduction of *conflicting interactions* [14–31], e.g., collective pursuit–evasion (PE). PE is first formulated by a one-to-one model [18–21], which generally describes a scenario that, a pursuer tries to approach an evader *as near as possible* in order to capture it, while the evader, oppositely, tries to escape from the pursuer; and with years of development, it has been studied with many scenarios, e.g., one-pursuer–multiple-evaders [22–23], multiple-pursuers–one-evader [22–24], and multiple-pursuers–multiple-evaders with particular interest from statistical physics [14,15,27–30]. Among which, there are three types of descriptions: the kinematics/dynamics of agents described by ordinary differential equations [14,16,20–22,26,27]; the behaviors of agents described by discrete iterative equations [17,24]; the evolution of agents that restricted on discrete lattice [25,28–30].

Motivated by while different from PE, our investigation also falls into the category of agents with conflicting interactions, but focuses on a new and more complex type: defense–intrusion (DI) interaction, which is also ubiquitous and fascinating both in animal world (e.g., a prey mother tries to protect her child against a predator or predators) and artificial world (e.g., a guarded vessel tries to protect an island by preventing approach of opposed vessels). Compared with PE interactions, in DI

* Corresponding author.
E-mail address: LW@nju.edu.cn (W. Li).

http://dx.doi.org/10.1016/j.cnsns.2015.03.013
1007-5704/

interactions, intruders not merely escape from defenders but with additional intention to approach the vicinity of a protected region (or target), while defenders not merely pursue but with additional aim to expel intruders away from the target, which is rarely investigated. As a first step, we consider one-defender–one-intruder formulation, since (1) such system with merely two agents already shows very rich and even unexpected properties that needs a separate analysis; (2) further, the deep understanding of the system is a good starting point with important insights for considering DI interactions of multiple agents; and (3) the system has distinct features that are different from one-pursuer–one-evader with PE interaction that will add new knowledge of such games of two players.

In our formulation of the DI interaction, the intruder tries to approach a protected stationary target with avoidance of the defender, while the defender tries to expel the intruder away from the target as maximum as possible. Naturally, the strategies of the agents are modelled using artificial anti-Newtonian forces [31]; here the intruder is attracted by the target but repulsed by the defender; while the defender tries to approach an appropriate IP between the intruder and the target in order to intercept the intruder.

Intuitionally, to keep the intruder away from the target as maximum as possible, *one may wonder that:* is it the better strategy for the defender trying to approach the intruder as near as possible (i.e., selecting the IP very near the intruder), just as behaviors of pursuers in PE? *Unexpectedly, this is not always the case.*

To characterize the optimal strategy of the defender, we first introduce the *flexibility* for IP selection, and then investigate the stable motion patterns of the system (i.e., the *synchronous–concentric–circling pattern* and the *stationary confrontation pattern*), and characterize a transition of the motion patterns using the *phase-transition surface* for different attraction/repulsion coefficients. Then the optimal interception strategy of the defender can be interpreted from two perspectives (i.e., the *motion-pattern–transition* perspective and *relative-mobility* perspective), with distinct physical meanings. To be specific, (1) the optimal strategy of the defender is just the critical condition that renders the transition between the two stable motion patterns; (2) the optimal strategy just depends on relative mobility of two agents, e.g., when the defender's mobility is faster than the intruder's mobility, then the defender has more capability to behave more aggressively (i.e., choosing the IP very near the intruder); else, it is wise for the defender to behave conservatively; as a specific case, when the two agents have same mobility, it is optimal to choose the IP just as the midpoint between the intruder and the target.

Analysis and quantitative simulations provide important insights into the dynamics and the defense strategy, which are of interest for many disciplines, from biology and physics, to applied science such as control and robotics. And the results provide a good starting point with rich insights for considering collective motion of multi-agents with defense–intrusion interactions.

2. Problem formulation

2.1. Model

Consider two agents, one defender and one intruder, in the 2-dimensional Euclidean space, with positions of $p_1(t) \in \mathbb{R}^2$ and $p_a(t) \in \mathbb{R}^2$, respectively. The defender tries to protect a target, which is stationary and located at p_0, while the intruder tries to reach the vicinity of the target with avoidance of the defender, as illustrated in Fig. 1(a). Consider the kinematics of the agents described by two nonlinear ordinary differential equations:

$$\dot{p}_a(t) = u_a(t) \in \mathbb{R}^2, \quad \dot{p}_1(t) = u_1(t) \in \mathbb{R}^2,$$

where $u_1(t), u_a(t)$ are the strategies of the defender and the intruder, respectively. For the intruder, it is natural to assume that there is a virtual attraction from the target and a virtual repulsion from the defender, thus the feedback strategy of the intruder is:

$$u_a(t) = u_a^{att}(t) + u_a^{rep}(t), \tag{1}$$

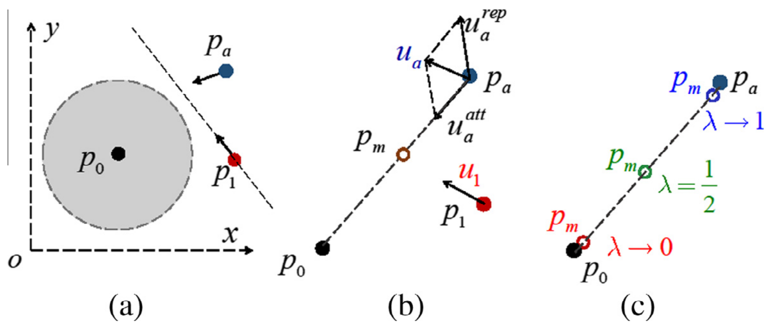

Fig. 1. (a) Illustration of the DI, the arrows denote possible moving directions of agents. (b) Illustration of forces $u_a(t)$, $u_a^{att}(t)$, $u_a^{rep}(t)$ and $u_1(t)$. (c) $p_m(t)$ as a function of λ to characterize the selection flexibility. For clarity, the parameter t of the notations in the figures is omitted.

where

$$u_a^{att}(t) = -k_a(p_a(t) - p_0), \quad u_a^{rep}(t) = k_r \frac{p_a(t) - p_1(t)}{\|p_a(t) - p_1(t)\|^2},$$

$k_a, k_r > 0$ are the attraction and repulsion coefficients, respectively, the term $u_a^{att}(t)$ models the attraction of the target, and $u_a^{rep}(t)$ is the repulsion from the defender, which increases as the distance between agents shrinks.

To intercept the intruder, the defender tries to move to an IP $p_m(t)$ between the intruder and the target, as illustrated in Fig. 1(b), thus the feedback strategy of the defender is:

$$u_1(t) = -k_1(p_1(t) - p_m(t)), \tag{2}$$

where $k_1 > 0$. The coefficients k_a and k_1 can be viewed as a certain type mobility (or the negative-feedback gains) of the defender and the intruder, respectively. Then naturally there is a question: to keep the intruder away from the target as maximum as possible, is it a better strategy for the defender trying to approach the intruder as near as possible, i.e., selecting $p_m(t) \to p_a(t)$? Unexpectedly, this is not always the case.

2.2. Flexibility of IP selection

To characterize the *flexibility* for selection of $p_m(t)$, we use a parameter λ, and

$$p_m(t) = \lambda p_a(t) + (1 - \lambda)p_0, \quad \lambda \in (0, 1],$$

as illustrated in Fig. 1(c), in which for example, (1) when, $\lambda \to 0$, $p_m(t) \to p_0$; (2) when $\lambda = 1$, $p_m(t) = p_a(t)$; (3) as a special case, for $\lambda = 0.5$, then $p_m(t)$ is just the midpoint between p_0 and $p_a(t)$.

Initially, the two agents and the target are in non-collinear configuration; also refer to Remark 1 in Section 3.1.

3. Analysis

3.1. Stable motion patterns

There are two stable motion patterns: the *synchronous–concentric–circling pattern*, for abbreviation, Pattern I [i.e., the agents move around the target with the same angular speed on two concentric circles respectively, refer to Fig. 2(a) and (b) for illustration] and the *stationary confrontation pattern*, for abbreviation, Pattern II [i.e., the agents reach a stationary collinear configuration, refer to Fig. 2(c) and (d) for illustration]. The initial condition has no influence on the stable motion patterns.

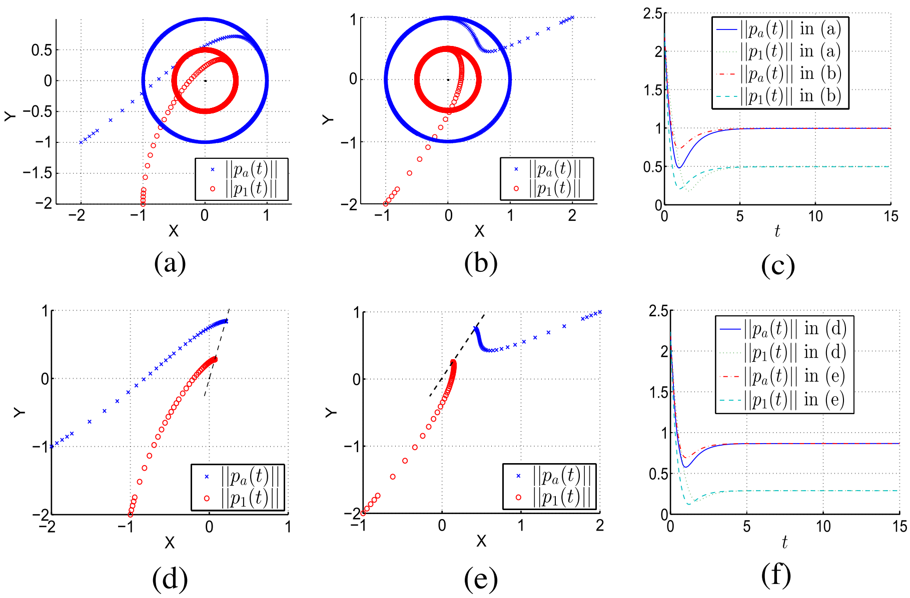

Fig. 2. Illustration of two stable motion patterns in 2D X–Y plane. (a)–(c) illustrate Pattern I, (e)–(f) illustrate Pattern II. $p_1(0) = (-1, -2)^T$, $k_a = 2$, $k_r = 1$, $k_1 = 2$. Without loss of generality, the target is located at the origin of the Cartesian Coordinates, i.e., $p_0 = [0, 0]^T$. (a) $\lambda = 0.5$, $p_a(0) = (-2, -1)^T$. (b) $\lambda = 0.5$, $p_a(0) = (2, 1)^T$. Initial positions have no effect on the stable motion patterns, except for the clockwise circling (a) or the counterclockwise circling (b). (d) $\lambda = 1/3$, $p_a(0) = (-2, -1)^T$. (e) $\lambda = 1/3$, $p_a(0) = (2, 1)^T$. Initial positions have no influence on values of $\|p_a(t)\||_{t\to\infty}$ and $\|p_1(t)\||_{t\to\infty}$.

Remark 1. In this paper, we do not consider the initial collinear configuration or any further actions of agents when in the states of the stable motion patterns. For example, in Pattern II, one may think that if the defender takes further action to approach the intruder, then the intruder will be expelled further away. This is indeed the case, but here the intruder just behaves as an evader, which violates the scenario of DI. If further actions considered, the intruder will naturally adopt a new strategy to approach the target while the defender will also take a new strategy, which is out of the scoop of this paper and leads to another topic for future investigation.

3.2. Phase-transition surface

For different parameters, the system may show different stable motion patterns, and there is a phase-transition as the parameters continuously change. Interestingly, this can be characterized elegantly by a *phase-transition surface* that defined as:

$$\Gamma(\lambda, k_1, k_a) := \lambda - \frac{k_1}{k_1 + k_a} = 0 \tag{3}$$

and Patterns I and II can be discriminated by the value of λ:

$$\begin{cases} \lambda > \dfrac{k_1}{k_1 + k_a}, & \text{Pattern I} \\ \lambda \leq \dfrac{k_1}{k_1 + k_a}, & \text{Pattern II} \end{cases} \tag{4}$$

as illustrated in Fig. 3, for the analytic derivation, refer to Appendix A; and for numerical simulations, refer to Figs. 4–6.

3.3. Optimal IP

Then what is the optimal IP of $p_m(t)$, or equivalently, the optimal value of λ? To answer this question, first denote $d_a = \lim_{t\to\infty} \|p_a(t)\|$ as the stable distance from the intruder to the target, which is a function of parameters λ, k_a, k_r, k_1, i.e., $d_a = d_a(\lambda, k_a, k_r, k_1)$, we use d_a for abbreviation if without confusion. The defender tries to maximize d_a by selecting an appropriate λ. For given k_a, k_r, k_1, the distance d_a has its maximum for certain value of λ (Figs. 4–6), i.e.,

$$\max_{\lambda\in(0,1]} d_a(\lambda, k_a, k_r, k_1) = d_a(\lambda_o, k_a, k_r, k_1), \tag{5}$$

where λ_o is defined as the optimal value of λ.

Distance d_a is an important criterion to evaluate the performance of the defender, which is a piecewise, continuous function (for derivation, refer to Appendix A) as follows:

$$d_a = \begin{cases} \sqrt{\dfrac{k_r}{\lambda k_1 - k_1 + k_a}}, & \lambda > \dfrac{k_1}{k_1 + k_a} \\ \sqrt{\dfrac{k_r}{(1-\lambda)k_a}}, & \lambda \leqslant \dfrac{k_1}{k_1 + k_a} \end{cases}. \tag{6}$$

From Eqs. (4) and (5), we get the optimal value λ_o:

$$\lambda_o = \frac{k_1}{k_1 + k_a}. \tag{7}$$

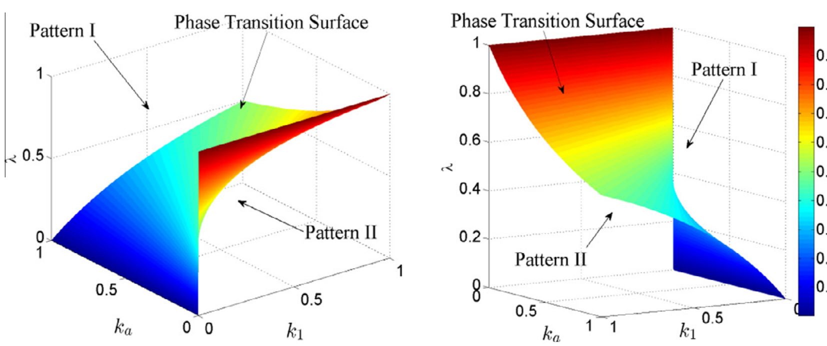

Fig. 3. Illustration of the phase-transition surface from two different view angles.

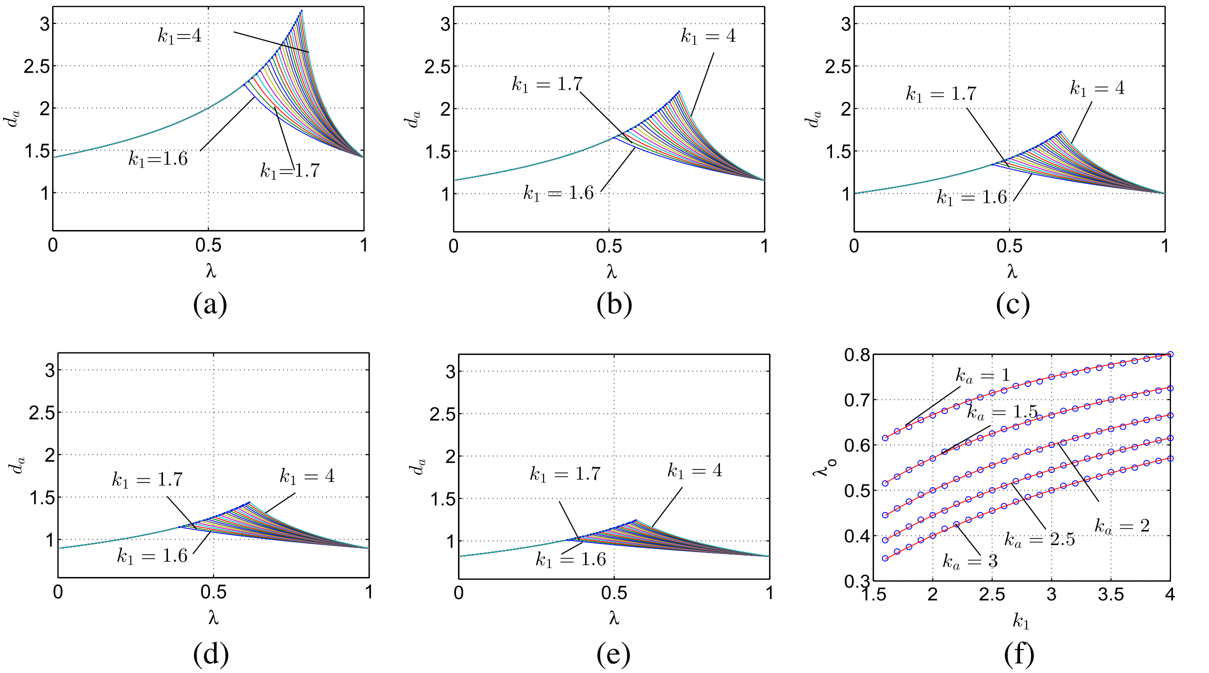

Fig. 4. Illustration of d_a as a function of λ, k_a, k_1, with k_1 ranging from 1.6 to 4, and k_a from 1 to 3, in (a)–(e) via simulation. $k_r = 2$, $p_1(0) = (-1, -2)^T$, $p_a(0) = (-2, -1)^T$. (a) $k_a = 1$. (b) $k_a = 1.5$. (c) $k_a = 2$. (d) $k_a = 2.5$. (e) $k_a = 3$. (f) The optimal value λ_o as a function of k_1 and k_a that derived from (a)–(e), the circles in (f) are the simulation results while the lines are the theoretical calculations.

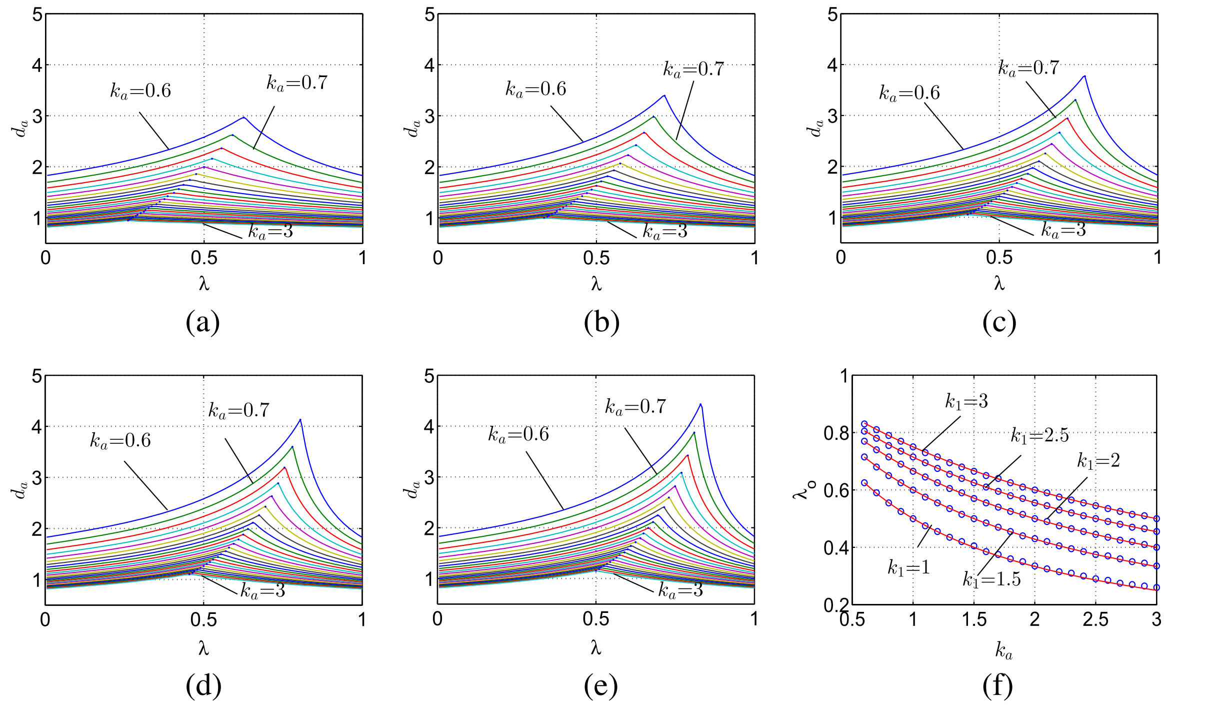

Fig. 5. Illustration of d_a as a function of λ, k_a, k_1, with k_a ranging from 0.6 to 3, and k_1 from 1 to 3, in (a)-(e) via simulation. $k_r = 2$, $p_1(0) = (-1, -2)^T$, $p_a(0) = (-2, -1)^T$. (a) $k_1 = 1$. (b) $k_1 = 1.5$. (c) $k_1 = 2$. (d) $k_1 = 2.5$. (e) $k_1 = 3$. (f) The optimal value λ_o as a function of k_1 and $k_a$that derived from (a)–(e), the circles in (f) are the simulation results while the lines are the theoretical calculations.

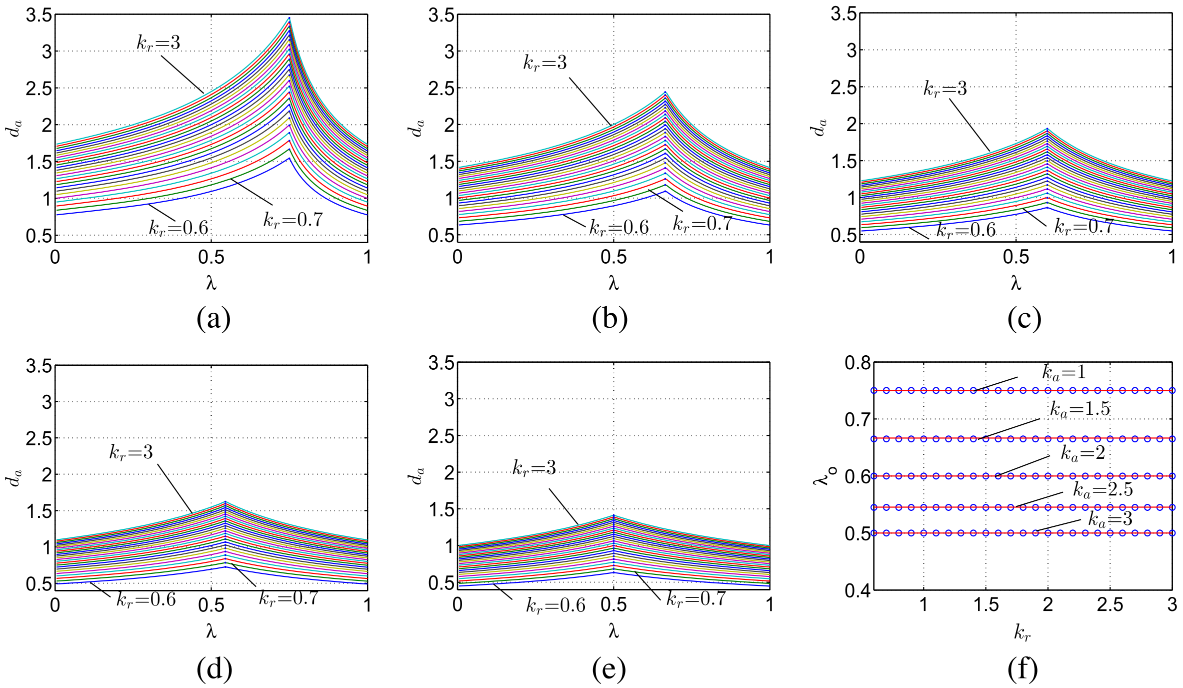

Fig. 6. Illustration of d_a as a function of λ, k_a, k_r,with k_r ranging from 0.6 to 3, and k_a from 1 to 3, in (a)–(e) via simulation. $k_1 = 3$, $p_1(0) = (-1, -2)^T$, $p_a(0) = (-2, -1)^T$. (a) $k_a = 1$. (b) $k_a = 1.5$. (c) $k_a = 2$. (d) $k_a = 2.5$. (e) $k_a = 3$ (f) the optimal value λ_o as a function of k_r and k_a that derived from (a)–(e), the circles in (f) are the simulation results while the lines are the theoretical calculations.

d_a increases in $\lambda \in (0, \lambda_o]$ and decreases in $\lambda \in [\lambda_o, 1]$. Thus when $\lambda = \lambda_o$,

$$\max_{\lambda\in(0,1]} d_a = d_a|_{\lambda=\lambda_o} = \sqrt{\frac{k_r(k_1+k_a)}{k_a^2}}.$$

Then compare Eq. (6) with the phase-transition surface described by Eq. (3), we can see that the optimal value λ_o is just located on the phase-transition surface. Pattern I shows that, the defender is over-aggressive, i.e., it is better to decrease λ to λ_o; and if Pattern II can be easily achieved, then it shows that, the defender is conservative, that is, the defender still has potential or capability to behave more aggressively (i.e., increase λ to λ_o).

From another perspective, the optimal strategy λ_o of the defender in Eq. (6) is a function of relative mobility of agents: for example, when the mobility of the defender is much larger (i.e., $k_1 > k_a$), it is optimal to select $\lambda_o \approx 1$; when the agents have the same mobility (i.e., $k_1 = k_a$), then trying to approach the midpoint between the target and the intruder (i.e., select $\lambda_o = 0.5$) is the optimal choice.

Interestingly, for Pattern I, we also have:

(1) no matter what value of λ, the distance between the intruder and the defender is always the constant value $\sqrt{k_r/(k_1+k_a)}$ [refer to Eq. (A7)], and
(2) when $\lambda \to \lambda_o$, then from Eq. (A8), $v_1/v_a \to \lambda_o$, where v_a and v_1 are denoted as the stable speeds of the intruder and the defender, respectively.

4. Conclusion

In this paper, we provide and characterize the DI model, analyze the optimal interception strategy for the defender, and illustrate two stable motion patterns with characterization of the phase-transition surface. Then we interpret the physical meaning of the optimal interception strategy from two perspectives. The results provide important insights into collective DI of multiple agents that will be investigated in future. The results will also provide insights into similar scenarios, for example, to derive efficient strategies for Area Persistent Denial by autonomous vehicles.

Acknowledgements

This work is funded by National Science Foundation of China, Grant No. 61473142.

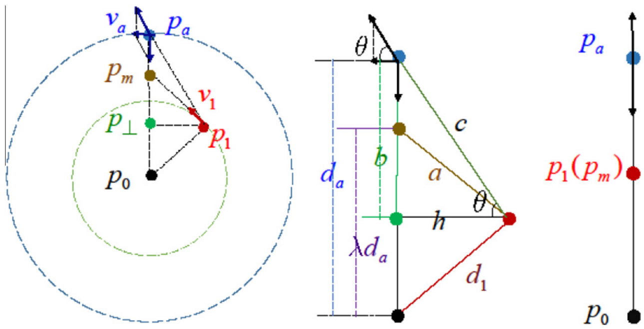

Fig. A. Illustration of Patten I (the left two figures) and Pattern II (the right figure). $p_\perp$ is the foot point of triangle $p_0p_1p_m$, h is the distance between p_1 and $p_\perp$, a is the distance between p_m and p_1; b is the distance between $p_\perp$ and p_a; c is the distance between p_a and p_1.

Appendix A

The two patterns and thus the phase-transition surface are derived as follows. First, Patten I (as illustrated in Fig. A) can be characterized by the following equations:

$$\begin{cases} \frac{v_a}{d_a} = \frac{v_1}{d_1}, & \text{(A1)} \\ v_1 = ak_1, & \text{(A2)} \\ v_a = \frac{k_r}{c}\cos\theta, & \text{(A3)} \\ d_a = \frac{k_r}{ck_a}\sin\theta, & \text{(A4)} \end{cases}$$

where Eq. (A1) holds since the two agents have the same angular velocity; Eq. (A2) is derived from Eq. (2); Eq. (A3) holds since

$$\|u_a^{rep}\| = \frac{k_r}{\|p_a - p_1\|} = \frac{k_r}{c} \quad \text{and} \quad v_a = \|u_a^{rep}\|\cos\theta;$$

and Eq. (A4) holds since

$$\|u_a^{att}\| = k_a d_a = \|u_a^{rep}\|\cos\theta = \frac{k_r}{c}\sin\theta,$$

in which the parameters a, c can be solved geometrically as follows:

$$\begin{cases} a = \sqrt{\lambda^2 d_a^2 - d_1^2}, \lambda^2 d_a^2 > d_1^2 \\ b = d_a - \sqrt{d_1^2 - h^2} = \frac{\lambda d_a^2 - d_1^2}{\lambda d_a} \\ c = \sqrt{h^2 + b^2} = \sqrt{\frac{\lambda d_1^2 + \lambda d_a^2 - 2d_1^2}{\lambda}} \\ h = \frac{ad_1}{\lambda d_a} \end{cases}.$$

Then, substitute Eqs. (A2)–(A4) to Eq. (A1) and note that $\cos\theta = h/c, \sin\theta = b/c$, then we have

$$k_a d_1^2 = \left(\lambda d_a^2 - d_1^2\right)k_1, \tag{A5}$$

i.e.,

$$d_1^2 = \frac{\lambda k_1}{k_1 + k_a} d_a^2. \tag{A6}$$

Note that $\lambda^2 d_a^2 > d_1^2$, for Pattern I, we get the constraint

$$\lambda > \frac{k_1}{k_1 + k_a}.$$

Substitute Eqs. (A5) and (A6) to Eq. (A4), we get:

$$d_a^2 = \frac{k_r}{\lambda k_1 - k_1 + k_a},$$
$$v_1 = k_1 d_a \sqrt{\frac{k_1\lambda^2 + k_a\lambda^2 - k_1\lambda}{k_1 + k_a}},$$
$$v_a = k_1 d_a \sqrt{\lambda - 1 + \frac{k_a}{k_1}\lambda},$$
$$c = \sqrt{\frac{k_r}{k_1 + k_a}}, \tag{A7}$$

$$\frac{v_1}{v_a} = \sqrt{\frac{k_1\lambda}{k_1 + k_a}}. \tag{A8}$$

For Pattern II, the position of the defender is just located at p_m, thus for the intruder, the repulsive force equals to the attractive force, i.e.,

$$k_a d_a = \frac{k_r}{(1-\lambda)d_a}.$$

That is:

$$d_a = \sqrt{\frac{k_r}{(1-\lambda)k_a}}, \quad \lambda \in \left(0\frac{k_1}{k_1 + k_a}\right].$$

For the phase-transition surface, note that, for Pattern I, $\lambda > \frac{k_1}{k_1+k_a}$, and for Pattern II, $\lambda \in \left(0\frac{k_1}{k_1+k_a}\right]$, the result holds.

References

[1] Farrell FDC, Marchetti MC, Marenduzzo D, Tailleur J. Pattern formation in self-propelled particles with density-dependent motility. Phys Rev Lett 2012;108:248101.
[2] Peshkov A, Aranson IS, Bertin E, Chaté H, Ginelli F. Nonlinear field equations for aligning self-propelled rods. Phys Rev Lett 2012;109:268701.
[3] Yamanaka S, Ohta T. Formation and collision of traveling bands in interacting deformable self-propelled particles. Phys Rev E 2014;89:012918.
[4] Couzin ID, Krause J, Franks NR, Levin SA. Effective leadership and decision-making in animal groups on the move. Nature 2005;433:513.
[5] Chaté H, Ginelli F, Grégoire G, Raynaud F. Collective motion of self-propelled particles interacting without cohesion. Phys Rev E 2008;77:046113.
[6] Ihle T. Invasion-wave-induced first-order phase transition in systems of active particles. Phys Rev E 2013;88:040303.
[7] Peshkov A, Ngo S, Bertin E, Chaté H, Ginelli F. Continuous theory of active matter systems with metric-free interactions. Phys Rev Lett 2012;109:098101.
[8] Vicsek T, Zafeiris A. Collective motion. Phys Rep 2012;517:71.
[9] Romensky M, Lobaskin V, Ihle T. Tricritical points in a Vicsek model of self-propelled particles with bounded confidence. Phys Rev E 2014;90:063315.
[10] Morin A, Caussin J, Eloy C, Bartolo D. Collective motion with anticipation: flocking, spinning, and swarming. Phys Rev E 2015;91:012134.
[11] Sato M. Chasing and escaping by three groups of species. Phys Rev E 2012;85:066102.
[12] Solon AP, Chate H, Tailleur J. From Phase to Microphase Separation in Flocking Models: The Essential Role of Nonequilibrium Fluctuations. Phys Rev Lett 2015;114:068101.
[13] Morin A et al. Collective motion with anticipation: Flocking, spinning, and swarming. Phys Rev E 2015;91:012134.
[14] Sierra DA, McCullough P, Olgac N, Adams E. Control of antagonistic swarm dynamics via Lyapunov's method. Asian J Control 2012;14(4):1–13.
[15] Sato M. Chasing and escaping by three groups of species. Phys Rev E 2012;85:066102.
[16] Romanczuk P, Couzin ID, Schimansky-Geier L. Collective motion due to individual escape and pursuit response. Phys Rev Lett 2009;102:01060.
[17] Angelani L. Collective predation and escape strategies. Phys Rev Lett 2012;109:118104.
[18] Isaacs R. Differential games. New York: Wiley; 1965.
[19] Nahin PJ. Chases and escapes: the mathematics of pursuit and evasion. Princeton: Princeton University Press; 2007.
[20] Behroozi F, Gagnon R. The goose chase. Am J Phys 1979;47:237.
[21] Mungan CE. A classic chase problem solved from a physics perspective. Eur J Phys 2005;26:985.
[22] Scott W, Leonard N. Pursuit, herding and evasion: a three-agent model of caribou predation. Proc Am Control Conf 2013;2978.
[23] Escobedo R, Muro C, Spector L, Coppinger RP. Group size, individual role differentiation and effectiveness of cooperation in a homogeneous group of hunters. J R Soc Interface 2014;11:20140204.
[24] Kopparty S, Ravishankar CV. A framework for pursuit evasion games in $\mathbb{R}^n$. Inf Process Lett 2005;96:114.
[25] Oshanin G, Vasilyev O, Krapvisky PL, Klafter JK. Survival of an evasive prey. Proc Natl Acad Sci USA 2009;106:13696.
[26] Chen Y, Kolokolnikov T. A minimal model of predator–swarm interactions. J R Soc Interface 2013;11:20131208.
[27] Zhdankin V, Sprott JC. Simple predator-prey swarming model. Phys Rev E 2010;82:056209.
[28] Iwama T, Sato M. Group chase and escape with some fast chasers. Phys Rev E 2012;86:067102.
[29] Kamimura A, Ohira T. Group chase and escape. New J Phys 2010;12:053013.
[30] Vicsek T. Closing in on evaders. Nature 2010;466:43.
[31] Sprott JC. Anti-Newtonian dynamics. Am J Phys 2009;77:783.